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Constants of t-dist^{n}: — Since $f(t)$ is symmetrical about the line $t=0$, all the moments of odd order about origin vanish i.e.

$$\mu'_{2r+1} \text{ (about origin)} = 0 = \cancel{\text{Mean}}$$
$$r = 0, 1, 2, \ldots$$

In particular $r=0$
$$\mu_1' = 0 = \text{Mean}$$

Hence central moments coincide with moments about origin
$$\mu_{2r+1} = 0, \quad (r = 1, 2, \ldots) \quad \text{—— (1)}$$

The moments of even order are given by

$$\mu_{2r} = \mu'_{2r} \text{ (about origin)}$$
$$= \int_{-\infty}^{\infty} t^{2r} f(t)\, dt = 2\int_0^{\infty} t^{2r} f(t)\, dt$$
$$= 2 \cdot \frac{1}{B\left(\frac{1}{2}, \frac{n}{2}\right)\sqrt{n}} \int_0^{\infty} \frac{t^{2r}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\, dt$$

The integral is absolutely convergent if $2r < n$

Put $1 + \frac{t^2}{n} = \frac{1}{y} \Rightarrow t^2 = \frac{n(1-y)}{y}$

i.e. $2t\, dt = \frac{-n}{y^2}\, dy$

when $t=0$, $y=1$ and when $t=\infty$, $y=0$.

Therefore

$$\mu_{2r} = \frac{2}{\sqrt{n}\, B\left(\frac{1}{2}, \frac{n}{2}\right)} \int_1^0 \frac{t^{2r}}{\left(\frac{1}{y}\right)^{\left(\frac{n+1}{2}\right)}} \frac{-n}{2ty^2}\, dy$$

$$= \frac{n}{\sqrt{n}\, B\left(\frac{1}{2}, \frac{n}{2}\right)} \int_0^1 (t^2)^{(2r-1)/2}\, y^{\left(\frac{n+1}{2}\right)-2}\, dy$$

$$= \frac{\sqrt{n}}{B\left(\frac{1}{2}, \frac{n}{2}\right)} \int_0^1 \left[n\left(\frac{1-y}{y}\right)\right]^{r-\frac{1}{2}} y^{(n+1)/2 - 2}\, dy$$

$$= \frac{n^r}{B\left(\frac{1}{2}, \frac{n}{2}\right)} \int_0^1 y^{\frac{n}{2}-r-1} (1-y)^{r-\frac{1}{2}}\, dy$$

$$\mu_{2r} = \frac{n^r}{B\left(\frac{1}{2}, \frac{n}{2}\right)} B\left(\frac{n}{2} - r,\ r + \frac{1}{2}\right);\ n > 2r \qquad (2)$$

$$= n^r \frac{\Gamma\left(\frac{n}{2} - r\right) \Gamma\left(r + \frac{1}{2}\right)}{\cancel{\Gamma\left(\frac{n+1}{2}\right)}} \times \frac{\cancel{\Gamma\left(\frac{n+1}{2}\right)}}{\Gamma\frac{1}{2}\, \Gamma\frac{n}{2}}$$

$$\mu_{2r} = \frac{n^r \left(r - \frac{1}{2}\right)\left(r - \frac{3}{2}\right) \cdots \frac{3}{2} \cdot \frac{1}{2}\, \cancel{\Gamma\frac{1}{2}}\, \cancel{\Gamma\left(\frac{n}{2} - r\right)}}{\cancel{\Gamma\frac{1}{2}} \left(\frac{n}{2} - 1\right)\left(\frac{n}{2} - 2\right) \cdots \left(\frac{n}{2} - r\right) \cancel{\Gamma\left(\frac{n}{2} - r\right)}}$$

$$\mu_{2r} = \frac{n^r \left(r-\frac{1}{2}\right)\left(r-\frac{3}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}}{\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\cdots\left(\frac{n}{2}-r\right)}$$

$$\mu_{2r} = \frac{n^r (2r-1)(2r-3)\cdots 3\cdot 1}{(n-2)(n-4)\cdots\cdots(n-2r)}, \quad \frac{n}{2} > r \qquad \text{—(3)}$$

In particular for $r=1$

$$\mu_2 = n\cdot\frac{1}{(n-2)} = \frac{n}{(n-2)}, \quad [n>2]$$

and $$\mu_4 = n^2\cdot\frac{3\cdot 1}{(n-2)(n-4)} = \frac{3n^2}{(n-2)(n-4)}, \quad [n>4]$$

Hence $$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0 \qquad \gamma_1 = 0 \text{ as } \gamma_1 = \sqrt{\beta_1}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3(n-2)}{(n-4)}$$

Remarks :-

1) as $n \to \infty$, $\beta_1 = 0$ and

$$\beta_2 = \lim_{n\to\infty} 3\left(\frac{n-2}{n-4}\right) = 3\lim_{n\to\infty}\left[\frac{1-\frac{2}{n}}{1-\frac{4}{n}}\right]$$

$$= 3$$

2) changing r to $(r-1)$ eq^{n} (3) dividing and simplifying, we shall get the recurrence relation for the moments as

$$\frac{\mu_{2r}}{\mu_{2r-2}} = \frac{n(2r-1)}{(n-2r)}, \quad \frac{n}{2} > r$$

3) Moment generating function i.e. $M_x(t)$ (m.g.f) of t-distⁿ does not exist as odd order moments does not exist.

We observe that

$t \sim t_n$

then all the moments of order $2r < n$ exist but the moments of order $2r \geq n$ do not exist

So m.g.f. does not exist.

Limiting Form of t-distribution :-

As $n \to \infty$ the p.d.f. of t-distⁿ with n d.f viz

$$f(t) = \frac{1}{\sqrt{n}\, B\left(\frac{1}{2}, \frac{n}{2}\right)} \left(1 + \frac{t^2}{n}\right)^{-\left(\frac{n+1}{2}\right)}.$$

$$\to \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \qquad -\infty < t < \infty$$

Proof:- $$\lim_{n\to\infty} \frac{1}{\sqrt{n}\, B\left(\frac{1}{2}, \frac{n}{2}\right)} = \lim \frac{1}{\sqrt{n}} \frac{\Gamma\frac{n+1}{2}}{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}}$$

$$= \frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{\pi}} \cdot \left(\frac{n}{2}\right)^{1/2} = \frac{1}{\sqrt{2\pi}}$$

$$\left[\because \Gamma\tfrac{1}{2} = \sqrt{\pi} \qquad \lim_{n\to\infty} \frac{\Gamma(n+k)}{\Gamma n} = n^k\right.$$

$$\therefore \lim_{n\to\infty} f(t) = \lim_{n\to\infty} \frac{1}{\sqrt{n}\, B\left(\frac{1}{2}, \frac{n}{2}\right)} \cdot \lim_{n\to\infty} \left[\left(1 + \frac{t^2}{n}\right)^n\right]^{\frac{1}{2}}$$

$$\times \lim_{n\to\infty} \left(1 + \frac{t^2}{n}\right)^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-t^2/2\right) \quad ; \quad -\infty < t < \infty$$

Hence for large d.f. t-distⁿ tends to standard normal distⁿ.

NOTE :— $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$

$\lim_{n\to\infty}\left(1+\frac{x}{a}\right)^n = 1$

$\int_{-\infty}^{\infty} e^{-a^2x^2}\,dx = \sqrt{\pi}/a$

$\sqrt{\pi} = \Gamma\frac{1}{2}$

$e^{-\log n} = e^{\log n^{-1}} = \frac{1}{n}$

$$B(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

$\int_0^{\infty} e^{-x} x^{n-1}\,dx = \Gamma n$

$$\int_0^{\infty} e^{-ax} x^{n-1}\,dx = \frac{\Gamma n}{a^n}$$

$\int_0^1 x^m (1-x)^n\,dx = \beta(m,n)$

$$\int_0^{\infty} \frac{x^m}{(1+x)^{m+n}}\,dx = \beta(m,n)$$